Xue and Yuzbasi *Fixed Point Theory and Applications* (2015) 2015:34
DOI 10.1186/s13663-015-0275-8

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theorems for solutions of the stationary Schrödinger equation on cones

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Abstract

The main aim of this paper is to study and establish some new coincidence point and common fixed point theorems for solutions of the stationary Schrödinger equation on cones. An interesting application is to investigate the existence and uniqueness for solutions of the Dirichlet problem with respect to the Schrödinger operator on cones and the growth property of them.

Keywords: stationary Schrödinger equation; Poisson-Sch integral; cone

1 Introduction and main results

Let $\mathbf{R}$ and $\mathbf{R}_+$ be the set of all real numbers and the set of all positive real numbers, respectively. We denote by $\mathbf{R}^n$ ($n \geq 2$) the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $P = (X, x_n)$, $X = (x_1, x_2, \dots, x_{n-1})$. The Euclidean distance of two points P and Q in $\mathbf{R}^n$ is denoted by $|P - Q|$. Also $|P - O|$ with the origin O of $\mathbf{R}^n$ is simply denoted by $|P|$. The boundary, the closure, and the complement of a set $\mathbf{S}$ in $\mathbf{R}^n$ are denoted by $\partial\mathbf{S}$, $\overline{\mathbf{S}}$, and $\mathbf{S}^c$, respectively.

For $P \in \mathbf{R}^n$ and $r > 0$, let $B(P, r)$ denote the open ball with center at P and radius r in $\mathbf{R}^n$.

We introduce a system of spherical coordinates (r, Θ), $\Theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \dots, x_{n-1}, x_n)$ by $x_n = r\cos\theta_1$.

The unit sphere and the upper half unit sphere in $\mathbf{R}^n$ are denoted by $\mathbf{S}^{n-1}$ and $\mathbf{S}_+^{n-1}$, respectively. For simplicity, a point $(1, \Theta)$ on $\mathbf{S}^{n-1}$ and the set $\{\Theta; (1, \Theta) \in \Omega\}$ for a set Ω, $\Omega \subset \mathbf{S}^{n-1}$, are often identified with Θ and Ω, respectively. By $C_n(\Omega)$, we denote the set $\mathbf{R}_+ \times \Omega$ in $\mathbf{R}^n$ with the domain Ω on $\mathbf{S}^{n-1}$ ($n \geq 2$). We call it a cone. Then T_n is a special cone obtained by putting $\Omega = \mathbf{S}_+^{n-1}$. We denote the sets $I \times \Omega$ and $I \times \partial\Omega$ with an interval on $\mathbf{R}$ by $C_n(\Omega; I)$ and $S_n(\Omega; I)$. By $S_n(\Omega; r)$ we denote $C_n(\Omega) \cap S_r$. By $S_n(\Omega)$ we denote $S_n(\Omega; (0, +\infty))$, which is $\partial C_n(\Omega) - \{O\}$.

We shall say that a set $E \subset C_n(\Omega)$ has a covering $\{r_j, R_j\}$ if there exists a sequence of balls $\{B_j\}$ with centers in $C_n(\Omega)$ such that $E \subset \bigcup_{j=0}^{\infty} B_j$, where r_j is the radius of B_j and R_j is the distance from the origin to the center of B_j.

Let $C_n(\Omega)$ be an arbitrary domain in $\mathbf{R}^n$ and $\mathscr{A}_a$ denote the class of nonnegative radial potentials $a(P)$, *i.e.* $0 \leq a(P) = a(r)$, $P = (r, \Theta) \in C_n(\Omega)$, such that $a \in L_{\mathrm{loc}}^b(C_n(\Omega))$ with some $b > n/2$ if $n \geq 4$ and with $b = 2$ if $n = 2$ or $n = 3$.

This article is devoted to the stationary Schrödinger equation

$$\mathrm{Sch}_a\, u(P) = -\Delta u(P) + a(P)u(P) = 0 \quad \text{for } P \in C_n(\Omega),$$

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where Δ is the Laplace operator and $a \in \mathscr{A}_a$. The class of these solution is denoted by $H(a,\Omega)$. Note that they are the (classical) harmonic functions on cones in the case $a = 0$. Under these assumptions the operator Sch_a can be extended in the usual way from the space $C_0^\infty(C_n(\Omega))$ to an essentially self-adjoint operator on $L^2(C_n(\Omega))$ (see [1], Chapter 13). We will denote it Sch_a as well. The latter has a Green-Sch function $G_\Omega^a(P,Q)$. Here $G_\Omega^a(P,Q)$ is positive on $C_n(\Omega)$ and its inner normal derivative $\partial G_\Omega^a(P,Q)/\partial n_Q \geq 0$, where $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$. We denote this derivative by $PI_\Omega^a(P,Q)$, which is called the Poisson-Sch kernel with respect to $C_n(\Omega)$.

For positive functions h_1 and h_2, we say that $h_1 \lesssim h_2$ if $h_1 \leq Mh_2$ for some constant $M > 0$. If $h_1 \lesssim h_2$ and $h_2 \lesssim h_1$, we say that $h_1 \approx h_2$.

Let Ω be a domain on $\mathbf{S}^{n-1}$ with smooth boundary. Consider the Dirichlet problem

$$(\Lambda_n + \lambda)\varphi = 0 \quad \text{on } \Omega,$$

$$\varphi = 0 \quad \text{on } \partial\Omega,$$

where Λ_n is the spherical part of the Laplace opera Δ_n

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{\Lambda_n}{r^2}.$$

We denote the least positive eigenvalue of this boundary value problem by λ and the normalized positive eigenfunction corresponding to λ by $\varphi(\Theta)$, $\int_\Omega \varphi^2(\Theta)\,dS_1 = 1$. In order to ensure the existence of λ and a smooth $\varphi(\Theta)$. We put a rather strong assumption on Ω: if $n \geq 3$, then Ω is a $C^{2,\alpha}$-domain ($0 < \alpha < 1$) on $\mathbf{S}^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (*e.g.* see [2], pp.88-89, for the definition of a $C^{2,\alpha}$-domain).

For any $(1,\Theta) \in \Omega$, we have (see [3], pp.7-8)

$$\varphi(\Theta) \approx \operatorname{dist}\bigl((1,\Theta), \partial C_n(\Omega)\bigr),$$

which yields

$$\delta(P) \approx r\varphi(\Theta), \tag{1.1}$$

where $P = (r,\Theta) \in C_n(\Omega)$ and $\delta(P) = \operatorname{dist}(P, \partial C_n(\Omega))$.

We consider solutions of an ordinary differential equation

$$-Q''(r) - \frac{n-1}{r}Q'(r) + \left(\frac{\lambda}{r^2} + a(r)\right)Q(r) = 0, \quad 0 < r < \infty. \tag{1.2}$$

It is well known (see, for example, [4]) that if the potential $a \in \mathscr{A}_a$, then (1.2) has a fundamental system of positive solutions $\{V, W\}$ such that V is nondecreasing with (see [5–7])

$$0 \leq V(0+) \leq V(r) \quad \text{as } r \to +\infty,$$

and W is monotonically decreasing with

$$+\infty = W(0+) > W(r) \searrow 0 \quad \text{as } r \to +\infty.$$

We will also consider the class $\mathscr{B}_a$, consisting of the potentials $a \in \mathscr{A}_a$ such that there the finite limit $\lim_{r\to\infty} r^2 a(r) = k \in [0,\infty)$ exists, and moreover, $r^{-1}|r^2 a(r) - k| \in L(1,\infty)$. If $a \in \mathscr{B}_a$, then the (sub-) superfunctions are continuous (see [8]).

In the rest of paper, we assume that $a \in \mathscr{B}_a$ and we shall suppress denotation of this assumption for simplicity.

Denote

$$\iota_k^{\pm} = \frac{2 - n \pm \sqrt{(n-2)^2 + 4(k+\lambda)}}{2},$$

then the solutions to (1.2) have the asymptotic (see [9])

$$V(r) \approx r^{\iota_k^+}, \qquad W(r) \approx r^{\iota_k^-}, \quad \text{as } r \to \infty. \tag{1.3}$$

We denote the Green-Sch potential with a positive measure ν on $C_n(\Omega)$ by

$$G_\Omega^a \nu(P) = \int_{C_n(\Omega)} G_\Omega^a(P,Q)\, d\nu(Q).$$

The Poisson-Sch integral $PI_\Omega^a \mu(P)$ (resp. $PI_\Omega^a[g](P)) \not\equiv +\infty$ ($P \in C_n(\Omega)$) of μ (resp. g) on $C_n(\Omega)$ is defined as follows:

$$PI_\Omega^a \mu(P) = \frac{1}{c_n} \int_{S_n(\Omega)} PI_\Omega^a(P,Q)\, d\mu(Q)$$
$$\left(\text{resp. } PI_\Omega^a[g](P) = \frac{1}{c_n} \int_{S_n(\Omega)} PI_\Omega^a(P,Q) g(Q)\, d\sigma_Q\right),$$

where

$$PI_\Omega^a(P,Q) = \frac{\partial G_\Omega^a(P,Q)}{\partial n_Q}, \qquad c_n = \begin{cases} 2\pi, & n = 2, \\ (n-2)s_n, & n \geq 3, \end{cases}$$

μ is a positive measure on $\partial C_n(\Omega)$ (resp. g is a continuous function on $\partial C_n(\Omega)$ and $d\sigma_Q$ is the surface area element on $S_n(\Omega)$) and $\partial/\partial n_Q$ denotes the differentiation at Q along the inward normal into $C_n(\Omega)$.

We define the positive measure μ' on $\mathbf{R}^n$ by

$$d\mu'(Q) = \begin{cases} t^{-1} W(t) \frac{\partial \varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q), & Q = (t,\Phi) \in S_n(\Omega;(1,+\infty)), \\ 0, & Q \in \mathbf{R}^n - S_n(\Omega;(1,+\infty)). \end{cases}$$

Remark 1 If $d\mu(Q) = |g(Q)|\, d\sigma_Q$ ($Q = (t,\Phi) \in S_n(\Omega)$), where $g(Q)$ is a continuous function on $\partial C_n(\Omega)$, then we have (see [10, 11])

$$d\mu''(Q) = \begin{cases} |g(Q)| t^{-1} W(t) \frac{\partial \varphi(\Phi)}{\partial n_\Phi}\, d\sigma_Q, & Q = (t,\Phi) \in S_n(\Omega;(1,+\infty)), \\ 0, & Q \in \mathbf{R}^n - S_n(\Omega;(1,+\infty)). \end{cases}$$

Let $\epsilon > 0$, $0 \leq \alpha \leq n$, and λ be any positive measure on $\mathbf{R}^n$ having finite total mass. For each $P = (r,\Theta) \in \mathbf{R}^n - \{O\}$, the maximal function $M(P;\lambda,\alpha)$ is defined by (see [12–15])

$$M(P;\lambda,\alpha) = \sup_{0<\rho<\frac{r}{2}} \lambda\big(B(P,\rho)\big) V(\rho) W(\rho) \rho^{\alpha-2}.$$

The set

$$\{P=(r,\Theta)\in\mathbf{R}^n-\{O\};M(P;\lambda,\alpha)V^{-1}(r)W^{-1}(r)r^{2-\alpha}>\epsilon\}$$

is denoted by $E(\epsilon;\lambda,\alpha)$.

As on cones, Qiao [16], Corollaries 2.1 and 2.2, have proved the following result. For similar results, we refer the reader to papers by Qiao and Deng (see [17, 18]).

Theorem A *Let g be a continuous function on $\partial C_n(\Omega)$ satisfying*

$$\int_{S_n(\Omega)}\frac{|g(t,\Phi)|}{1+r^{-\iota_0^-+1}}\,d\sigma_Q<\infty. \tag{1.4}$$

Then $PI_\Omega^0[g](P)\in H(0,\Omega)$ and

$$\lim_{r\to\infty,P\in C_n(\Omega)} r^{-\iota_0^+}\varphi^{n-1}(\Theta)PI_\Omega^0[g](P)=0 \quad \bigl(P=(r,\Theta)\in C_n(\Omega)\bigr). \tag{1.5}$$

Theorem B *Let g be a continuous function on $\partial C_n(\Omega)$ satisfying* (1.4). *Then the function $PI_\Omega^0[g](P)$ $(P=(r,\Theta))$ satisfies*

$$PI_\Omega^0[g]\in C^2\bigl(C_n(\Omega)\bigr)\cap C^0\bigl(\overline{C_n(\Omega)}\bigr),$$
$$PI_\Omega^0[g](P)\in H(0,\Omega),$$
$$PI_\Omega^0[g]=g \quad \text{on } \partial C_n(\Omega),$$

and (1.5) *holds*.

Now we state our first result.

Theorem 1 *Let ϵ be a sufficiently small positive number and μ be a positive measure on $\partial C_n(\Omega)$ such that*

$$PI_\Omega^a\mu(P)\not\equiv+\infty \quad \bigl(P=(r,\Theta)\in C_n(\Omega)\bigr).$$

Then there exists a covering $\{r_j,R_j\}$ of $E(\epsilon;\mu',n-\alpha)$ $(\subset C_n(\Omega))$ satisfying

$$\sum_{j=0}^{\infty}\left(\frac{r_j}{R_j}\right)^{2-\alpha}\frac{V(R_j)W(R_j)}{V(r_j)W(r_j)}<\infty, \tag{1.6}$$

such that

$$\lim_{r\to\infty,P\in C_n(\Omega)-E(\epsilon;\mu',n-\alpha)} V^{-1}(r)\varphi^{\alpha-1}(\Theta)PI_\Omega^a\mu(P)=0 \quad \bigl(P=(r,\Theta)\in C_n(\Omega)\bigr).$$

Corollary 1 *Let μ be a positive measure on $S_n(\Omega)$ such that $PI_\Omega^a\mu(P)\not\equiv+\infty$ $(P\in C_n(\Omega))$. Then for a sufficiently large L and a sufficiently small ϵ we have*

$$\bigl\{P\in C_n\bigl(\Omega;(L,+\infty)\bigr);PI_\Omega^a\mu(P)\geq V(r)\varphi^{1-\alpha}(\Theta)\bigr\}\subset E\bigl(\epsilon;\mu',n-\alpha\bigr).$$

From (1.3) and Remark 1 we know that the following result generalizes Theorem A in the case $d\mu(Q) = |g(Q)|\,d\sigma_Q$.

Corollary 2 *Let g be a continuous function on* $\partial C_n(\Omega)$ *satisfying*

$$\int_{S_n(\Omega)} \frac{1}{1 + tW^{-1}(t)}\,d\mu(Q) < \infty. \tag{1.7}$$

Then $PI^a_\Omega\mu(P) \in H(a,\Omega)$ *and*

$$\lim_{r\to\infty, P\in C_n(\Omega)} V^{-1}(r)\varphi^{n-1}(\Theta)PI^a_\Omega\mu(P) = 0 \quad \bigl(P = (r,\Theta) \in C_n(\Omega)\bigr).$$

Our next aim is concerned with the solutions of the Dirichlet problem for the Schrödinger operator Sch_a on $C_n(\Omega)$ and the growth property of them.

Theorem 2 *Let* α, ϵ *be defined as in Theorem* 1 *and g be a continuous function on* $\partial C_n(\Omega)$ *satisfying*

$$\int_1^\infty t^{-1}V^{-1}(t)\left(\int_{\partial\Omega} \bigl|g(t,\Phi)\bigr|\,d_{\sigma_\Phi}\right)dt < +\infty, \tag{1.8}$$

where d_{σ_Φ} *is the surface area element of* $\partial\Omega$ *at* $\Phi \in \partial\Omega$. *Then the function* $PI^a_\Omega[g](P)$ ($P = (r,\Theta)$) *satisfies*

$$PI^a_\Omega[g] \in C^2\bigl(C_n(\Omega)\bigr) \cap C^0\bigl(\overline{C_n(\Omega)}\bigr),$$
$$PI^a_\Omega[g] \in H(a,\Omega),$$
$$PI^a_\Omega[g] = g \quad \text{on } \partial C_n(\Omega),$$

and there exists a covering $\{r_j, R_j\}$ *of* $E(\epsilon;\mu'',\alpha)$ *satisfying* (1.5) *such that*

$$\lim_{r\to\infty, P\in C_n(\Omega)-E(\epsilon;\mu'',\alpha)} V^{-1}(r)\varphi^{\alpha-1}(\Theta)PI^a_\Omega[g](P) = 0. \tag{1.9}$$

Remark 2 In the case $a = 0$, (1.8) is equivalent to (1.4) from (1.3). In the case $\alpha = n$, (1.6) is a finite sum, then the set $E(\epsilon;\mu'',0)$ is a bounded set and (1.9) holds in $C_n(\Omega)$, which generalizes Theorem B.

2 Some lemmas

Lemma 1 (see [1], p.354)

$$PI^a_\Omega(P,Q) \approx t^{-1}V(t)W(r)\varphi(\Theta)\frac{\partial\varphi(\Phi)}{\partial n_\Phi} \tag{2.1}$$

$$\left(\textit{resp. } PI^a_\Omega(P,Q) \approx V(r)t^{-1}W(t)\varphi(\Theta)\frac{\partial\varphi(\Phi)}{\partial n_\Phi}\right), \tag{2.2}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ *and any* $Q = (t,\Phi) \in S_n(\Omega)$ *satisfying* $0 < \frac{t}{r} \le \frac{4}{5}$ (*resp.* $0 < \frac{r}{t} \le \frac{4}{5}$);

$$PI^0_\Omega(P,Q) \lesssim \frac{\varphi(\Theta)}{t^{n-1}}\frac{\partial\varphi(\Phi)}{\partial n_\Phi} + \frac{r\varphi(\Theta)}{|P-Q|^n}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}, \tag{2.3}$$

for any $P = (r,\Theta) \in C_n(\Omega)$ *and any* $Q = (t,\Phi) \in S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))$.

Lemma 2 *Let μ be a positive measure on $S_n(\Omega)$ such that there is a sequence of points $P_i = (r_i, \Theta_i) \in C_n(\Omega)$, $r_i \to +\infty$ $(i \to +\infty)$ satisfying $PI^a_\Omega\mu(P_i) < +\infty$ $(i = 1, 2, \ldots)$. Then for a positive number l,*

$$\int_{S_n(\Omega;(l,+\infty))} \frac{W(t)}{t} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q) < +\infty \tag{2.4}$$

and

$$\lim_{R\to+\infty} \frac{W(R)}{V(R)} \int_{S_n(\Omega;(0,R))} \frac{V(t)}{t} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q) = 0. \tag{2.5}$$

Proof Take a positive number l satisfying $P_1 = (r_1, \Theta_1) \in C_n(\Omega)$, $r_1 \leq \frac{4}{5}l$. Then from (2.2), we have

$$V(r_1)\varphi(\Theta_1) \int_{S_n(\Omega;(l,+\infty))} \frac{W(t)}{t} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q) \lesssim \int_{S_n(\Omega)} PI^a_\Omega(P,Q)\, d\mu(Q) < +\infty,$$

which gives (2.4). For any positive number ϵ, from (2.4), we can take a number R_ϵ such that

$$\int_{S_n(\Omega;(R_\epsilon,+\infty))} \frac{W(t)}{t} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q) < \frac{\epsilon}{2}.$$

If we take a point $P_i = (r_i, \Theta_i) \in C_n(\Omega)$, $r_i \geq \frac{5}{4}R_\epsilon$, then we have from (2.1)

$$W(r_i)\varphi(\Theta_i) \int_{S_n(\Omega;(0,R_\epsilon])} \frac{V(t)}{t} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q) \lesssim \int_{S_n(\Omega)} PI^a_\Omega(P,Q)\, d\mu(Q) < +\infty.$$

If R $(R > R_\epsilon)$ is sufficiently large, then

$$\begin{aligned}
&\frac{W(R)}{V(R)} \int_{S_n(\Omega;(0,R))} \frac{V(t)}{t} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q) \\
&\quad\lesssim \frac{W(R)}{V(R)} \int_{S_n(\Omega;(0,R_\epsilon])} \frac{V(t)}{t} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q) + \int_{S_n(\Omega;(R_\epsilon,R))} \frac{W(t)}{t} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q) \\
&\quad\lesssim \frac{W(R)}{V(R)} \int_{S_n(\Omega;(0,R_\epsilon])} \frac{V(t)}{t} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q) + \int_{S_n(\Omega;(R_\epsilon,+\infty))} \frac{W(t)}{t} \frac{\partial\varphi(\Phi)}{\partial n_\Phi}\, d\mu(Q) \\
&\quad\lesssim \epsilon,
\end{aligned}$$

which gives (2.5). □

Lemma 3 *Let $\epsilon > 0$, $0 \leq \alpha \leq n$ and λ be any positive measure on $\mathbf{R}^n$ having a finite total mass. Then $E(\epsilon; \lambda, \alpha)$ has a covering $\{r_j, R_j\}$ $(j = 1, 2, \ldots)$ satisfying*

$$\sum_{j=1}^{\infty} \left(\frac{r_j}{R_j}\right)^{2-\alpha} \frac{V(R_j)W(R_j)}{V(r_j)W(r_j)} < \infty.$$

Proof Set

$$E_j(\epsilon; \lambda, \beta) = \left\{P = (r, \Theta) \in E(\epsilon; \lambda, \beta) : 2^j \leq r < 2^{j+1}\right\} \quad (j = 2, 3, 4, \ldots).$$

If $P = (r,\Theta) \in E_j(\epsilon;\lambda,\beta)$, then there exists a positive number $\rho(P)$ such that

$$\left(\frac{\rho(P)}{r}\right)^{2-\alpha} \frac{V(r)W(R)}{V(\rho(P))W(\rho(P))} \approx \left(\frac{\rho(P)}{r}\right)^{n-\alpha} \leq \frac{\lambda(B(P,\rho(P)))}{\epsilon}.$$

Since $E_j(\epsilon;\lambda,\beta)$ can be covered by the union of a family of balls $\{B(P_{j,i},\rho_{j,i}) : P_{j,i} \in E_k(\epsilon;\lambda,\beta)\}$ ($\rho_{j,i} = \rho(P_{j,i})$). By the Vitali lemma (see [19]), there exists $\Lambda_j \subset E_j(\epsilon;\lambda,\beta)$, which is at most countable, such that $\{B(P_{j,i},\rho_{j,i}) : P_{j,i} \in \Lambda_j\}$ are disjoint and $E_j(\epsilon;\lambda,\beta) \subset \bigcup_{P_{j,i}\in\Lambda_j} B(P_{j,i},5\rho_{j,i})$.

Therefore

$$\bigcup_{j=2}^{\infty} E_j(\epsilon;\lambda,\beta) \subset \bigcup_{j=2}^{\infty} \bigcup_{P_{j,i}\in\Lambda_j} B(P_{j,i},5\rho_{j,i}).$$

On the other hand, note that $\bigcup_{P_{j,i}\in\Lambda_j} B(P_{j,i},\rho_{j,i}) \subset \{P = (r,\Theta) : 2^{j-1} \leq r < 2^{j+2}\}$, so that

$$\sum_{P_{j,i}\in\Lambda_j} \left(\frac{5\rho_{j,i}}{|P_{j,i}|}\right)^{2-\alpha} \frac{V(|P_{j,i}|)W(|P_{j,i}|)}{V(\rho_{j,i})W(\rho_{j,i})} \leq \frac{5^{n-\alpha}}{\epsilon}\lambda\bigl(C_n\bigl(\Omega;\bigl[2^{j-1},2^{j+2}\bigr)\bigr)\bigr).$$

Hence we obtain

$$\begin{aligned}
\sum_{j=1}^{\infty} \sum_{P_{j,i}\in\Lambda_j} \left(\frac{\rho_{j,i}}{|P_{j,i}|}\right)^{2-\alpha} \frac{V(|P_{j,i}|)W(|P_{j,i}|)}{V(\rho_{j,i})W(\rho_{j,i})} &\approx \sum_{j=1}^{\infty} \sum_{P_{j,i}\in\Lambda_j} \left(\frac{\rho_{j,i}}{|P_{j,i}|}\right)^{n-\alpha} \\
&\leq \sum_{j=1}^{\infty} \frac{\lambda(C_n(\Omega;[2^{j-1},2^{j+2})))}{\epsilon} \\
&\leq \frac{3\lambda(\mathbf{R}^n)}{\epsilon}.
\end{aligned}$$

Since $E(\epsilon;\lambda,\beta) \cap \{P = (r,\Theta) \in \mathbf{R}^n; r \geq 4\} = \bigcup_{j=2}^{\infty} E_j(\epsilon;\lambda,\beta)$. Then $E(\epsilon;\lambda,\beta)$ is finally covered by a sequence of balls $\{B(P_{j,i},\rho_{j,i}), B(P_1,6)\}$ ($j = 2,3,\ldots$; $i = 1,2,\ldots$) satisfying

$$\sum_{j,i} \left(\frac{\rho_{j,i}}{|P_{j,i}|}\right)^{2-\alpha} \frac{V(|P_{j,i}|)W(|P_{j,i}|)}{V(\rho_{j,i})W(\rho_{j,i})} \approx \sum_{j,i} \left(\frac{\rho_{j,i}}{|P_{j,i}|}\right)^{n-\alpha} \leq \frac{3\lambda(\mathbf{R}^n)}{\epsilon} + 6^{n-\alpha} < +\infty,$$

where $B(P_1,6)$ ($P_1 = (1,0,\ldots,0) \in \mathbf{R}^n$) is the ball which covers $\{P = (r,\Theta) \in \mathbf{R}^n; r < 4\}$. □

3 Proof of Theorem 1

Take any point $P = (r,\Theta) \in C_n(\Omega;(R,+\infty)) - E(\epsilon;\mu',\alpha)$, where R ($\leq \frac{4}{5}r$) is a sufficiently large number and ϵ is a sufficiently small positive number.

Write

$$PI_\Omega^a \mu(P) = PI_\Omega^a(1)(P) + PI_\Omega^a(2)(P) + PI_\Omega^a(3)(P),$$

where

$$PI_\Omega^a(1)(P) = \frac{1}{c_n} \int_{S_n(\Omega;(0,\frac{4}{5}r])} PI_\Omega^a(P,Q)\,d\mu(Q),$$

$$PI^a_\Omega(2)(P) = \frac{1}{c_n}\int_{S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} PI^a_\Omega(P,Q)\,d\mu(Q),$$

and

$$PI^a_\Omega(3)(P) = \frac{1}{c_n}\int_{S_n(\Omega;[\frac{5}{4}r,\infty))} PI^a_\Omega(P,Q)\,d\mu(Q).$$

The relation $G^a_\Omega(P,Q) \le G^0_\Omega(P,Q)$ implies this inequality (see [20])

$$PI^a_\Omega(P,Q) \le PI^0_\Omega(P,Q). \tag{3.1}$$

By (2.1), (2.2), and Lemma 2, we have the following growth estimates:

$$PI^a_\Omega(1)(P) \lesssim V(r)\varphi(\Theta)\frac{W(\frac{4}{5}r)}{V(\frac{4}{5}r)}\int_{S_n(\Omega;(0,\frac{4}{5}r])}\frac{V(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}\,d\mu(Q) \lesssim \epsilon V(r)\varphi(\Theta), \tag{3.2}$$

$$PI^a_\Omega(3)(P) \lesssim V(r)\varphi(\Theta)\int_{S_n(\Omega;[\frac{5}{4}r,\infty))}\frac{W(t)}{t}\frac{\partial\varphi(\Phi)}{\partial n_\Phi}\,d\mu(Q) \lesssim \epsilon V(r)\varphi(\Theta). \tag{3.3}$$

By (3.1) and (2.3), we write

$$PI^a_\Omega(2)(P) \lesssim PI^a_\Omega(21)(P) + PI^a_\Omega(22)(P),$$

where

$$PI^a_\Omega(21)(P) = \int_{S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} V(t)\varphi(\Theta)\,d\mu'(Q)$$

and

$$PI^a_\Omega(22)(P) = \int_{S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r))} \frac{\operatorname{tr}\varphi(\Theta)}{|P-Q|^n W(t)}\,d\mu'(Q).$$

We first have

$$PI^a_\Omega(21)(P) \lesssim \epsilon V(r)\varphi(\Theta) \tag{3.4}$$

from Lemma 2.

Next, we shall estimate $PI^a_\Omega(22)(P)$. Take a sufficiently small positive number c such that $S_n(\Omega;(\frac{4}{5}r,\frac{5}{4}r)) \subset B(P,\frac{1}{2}r)$ for any $P=(r,\Theta)\in\Lambda(c)$, where

$$\Lambda(c) = \left\{P=(r,\Theta)\in C_n(\Omega);\ \inf_{z\in\partial\Omega}\bigl|(1,\Theta)-(1,z)\bigr| < c, 0<r<\infty\right\},$$

and divide $C_n(\Omega)$ into two sets $\Lambda(c)$ and $C_n(\Omega)-\Lambda(c)$.

If $P=(r,\Theta)\in C_n(\Omega)-\Lambda(c)$, then there exists a positive c' such that $|P-Q|\ge c'r$ for any $Q\in S_n(\Omega)$, and hence

$$PI^a_\Omega(22)(P) \lesssim \epsilon V(r)\varphi(\Theta) \tag{3.5}$$

from Lemma 2.

We shall consider the case $P \in \Lambda(c)$. Now put

$$H_i(P) = \left\{Q \in S_n\left(\Omega; \left(\frac{4}{5}r, \frac{5}{4}r\right)\right); 2^{i-1}\delta(P) \leq |P - Q| < 2^i\delta(P)\right\}.$$

Since $S_n(\Omega) \cap \{Q \in \mathbf{R}^n : |P - Q| < \delta(P)\} = \varnothing$, we have

$$PI^a_\Omega(22)(P) = \sum_{i=1}^{i(P)} \int_{H_i(P)} \frac{\operatorname{tr}\varphi(\Theta)}{|P - Q|^n W(t)}\, d\mu'(Q),$$

where $i(P)$ is a positive integer satisfying $2^{i(P)-1}\delta(P) \leq \frac{r}{2} < 2^{i(P)}\delta(P)$.

By (1.1) we have $r\varphi(\Theta) \lesssim \delta(P)$ $(P = (r, \Theta) \in C_n(\Omega))$, and hence

$$\int_{H_i(P)} \frac{\operatorname{tr}\varphi(\Theta)}{|P - Q|^n W(t)}\, d\mu'(Q) \lesssim \frac{r^{2-\alpha}}{W(r)}\varphi^{1-\alpha}(\Theta)\frac{\mu'(H_i(P))}{\{2^i\delta(P)\}^{n-\alpha}}$$

for $i = 0, 1, 2, \ldots, i(P)$.

Since $P = (r, \Theta) \notin E(\epsilon; \mu', \alpha)$, we have from (1.3)

$$\begin{aligned}\frac{\mu'(H_i(P))}{\{2^i\delta(P)\}^{n-\alpha}} &\lesssim \mu'\bigl(B\bigl(P, 2^i\delta(P)\bigr)\bigr)V\bigl(2^i\delta(P)\bigr)W\bigl(2^i\delta(P)\bigr)\bigl\{2^i\delta(P)\bigr\}^{\alpha-2} \\ &\lesssim M\bigl(P; \mu', \alpha\bigr) \\ &\leq \epsilon\epsilon V(r)W(r)r^{\alpha-2} \quad \bigl(i = 0, 1, 2, \ldots, i(P) - 1\bigr)\end{aligned}$$

and

$$\frac{\mu'(H_{i(P)}(P))}{\{2^i\delta(P)\}^{\alpha}} \lesssim \mu'\left(B\left(P, \frac{r}{2}\right)\right)V\left(\frac{r}{2}\right)W\left(\frac{r}{2}\right)\left(\frac{r}{2}\right)^{\alpha-2} \leq \epsilon V(r)W(r)r^{\alpha-2}.$$

So

$$PI^a_\Omega(22)(P) \lesssim \epsilon V(r)\varphi^{1-\alpha}(\Theta). \tag{3.6}$$

Combining (3.2)-(3.6), we finally find that if L is sufficiently large and ϵ is sufficiently small, then $PI^a_\Omega\mu(P) = o(V(r)\varphi^{1-\alpha}(\Theta))$ as $r \to \infty$, where $P = (r, \Theta) \in C_n(\Omega; (R, +\infty)) - E(\epsilon; \mu', \alpha)$. Finally, there exists an additional finite ball B_0 covering $C_n(\Omega; (0, R])$, which, together with Lemma 3, gives the conclusion of Theorem 1.

4 Proof of Theorem 2

For any fixed $P = (r, \Theta) \in C_n(\Omega)$, take a number R satisfying $R > \max(1, \frac{5}{4}r)$. By (1.7) and (2.2), we have

$$\begin{aligned}&\frac{1}{c_n}\int_{S_n(\Omega;(R,+\infty))} PI^a_\Omega(P, Q)\bigl|g(Q)\bigr|\, d\sigma_Q \\ &\quad\lesssim V(r)\varphi(\Theta)\int_R^\infty t^{-1}V^{-1}(t)\left(\int_{\partial\Omega}\bigl|g(t, \Phi)\bigr|\, d_{\sigma_\Phi}\right)dt < \infty.\end{aligned}$$

Thus $PI^a_\Omega[g](P)$ is finite for any $P \in C_n(\Omega)$. Since $PI^a_\Omega(P, Q) \in H(a, \Omega) \in H(a, \Omega)$ for any $Q \in S_n(\Omega)$, $PI^a_\Omega[g](P) \in H(a, \Omega)$.

Now we study the boundary behavior of $PI^a_\Omega[g](P)$. Let $Q' = (t', \Phi') \in \partial C_n(\Omega)$ be any fixed point and L be any positive number such that $L > \max\{t' + 1, \frac{4}{5}R\}$.

Set $\chi_{S(L)}$ is the characteristic function of $S(L) = \{Q = (t, \Phi) \in \partial C_n(\Omega), t \leq L\}$ and write

$$PI^a_\Omega[g](P) = PI^a_\Omega(1)[g](P) + PI^a_\Omega(2)[g](P),$$

where

$$PI^a_\Omega(1)[g](P) = \frac{1}{c_n}\int_{S_n(\Omega;(0,\frac{5}{4}L])} PI^a_\Omega(P,Q)g(Q)\,d\sigma_Q$$

and

$$PI^a_\Omega(2)[g](P) = \frac{1}{c_n}\int_{S_n(\Omega;(\frac{5}{4}L,\infty))} P^a_\Omega(P,Q)g(Q)\,d\sigma_Q.$$

Notice that $PI^a_\Omega(1)[g](P)$ is the Poisson-Sch integral of $g(Q)\chi_{S(\frac{5}{4}L)}$, we have

$$\lim_{P\to Q', P\in C_n(\Omega)} PI^a_\Omega(1)[g](P) = g\bigl(Q'\bigr).$$

Since $\lim_{\Theta\to\Phi'}\varphi(\Theta) = 0$, $PI^a_\Omega(2)[g](P) = O(V(r)\varphi(\Theta))$, and therefore tends to zero. So the function $PI^a_\Omega[g](P)$ can be continuously extended to $\overline{C_n(\Omega)}$ such that

$$\lim_{P\to Q', P\in C_n(\Omega)} PI^a_\Omega[g](P) = g\bigl(Q'\bigr)$$

for any $Q' = (t', \Phi') \in \partial C_n(\Omega)$ from the arbitrariness of L. Further, (1.9) is the conclusion of Theorem 1. Thus we complete the proof of Theorem 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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Acknowledgements
The last half of this work was done while the second author stayed at Istanbul University during the program 'Nonlinear Partial Differential Equations'. We are grateful to Istanbul University and the steering committee of the program. The authors are very thankful to the anonymous referees for their valuable comments and constructive suggestions, which helped to improve the quality of the paper.

Received: 29 November 2014 Accepted: 29 January 2015 Published online: 27 February 2015

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